A Lecture on Stereographic Projection and Inversion

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1 Stereographic projection

Let $\mathbb{E}^3$ denote the three dimensional euclidean space and $S = S_R(O)$ be a sphere in $\mathbb{E}^3$ with center O and radius R, i.e.

$$S_R(O) := \{p \in \mathbb{E}^3 \, s.t. \, |p - O| = R\}.$$

Let $\Pi \subset \mathbb{E}^3$ be a 2-dimensional plane passing through O and let $r \subset \mathbb{E}^3$ be the straight line passing through O and perpendicular to Π. Let p_N, p_S be the intersection points of r with S and call p_N, p_S the north and the south poles respectively.

Definition 1.1 (Stereographic projection)**.** We define the *stereographic projection* $F : S \setminus \{p_N\} \to \Pi$ as follows: for any $A \in S$ let $p_N A \subset \mathbb{E}^3$ be the unique straight line passing through A and p_N, and define $F(A) \in \Pi$ to be the unique intersection point of $p_N A$ with Π.

Exercise 1.2. [1]

1. *If $\ell \subset S$ is a great circle passing through p_N then $F(\ell)$ is a straight line in Π passing through O.*
2. *If $\ell = S \cap \Pi$ then $F(\ell) = \ell$.*
3. *Show that F is not an isometry, i.e. give an example of two points $A, B \in S$ so that $\mathrm{d}_S(A, B) \neq \mathrm{d}_\Pi(F(A), F(B))$.*
4. *$F(p_S) = O \in \Pi$ and if $A \to p_N$ then $F(A) \to \infty$.*
5. *If $\ell \subset S$ is any great circle not passing through p_N then $F(\ell) \subset \Pi$ is a circle in the plane.*
6. *if $\ell_1, \ell_2 \subset S$ are great circles in S and $A \in \ell_1 \cap \ell_2$, then*

$$\angle^S \ell_1 \, A \, \ell_2 = \angle^\Pi F(\ell_1) \, F(A) \, F(\ell_2),$$

where $\angle^S \ell_1 \, A \, \ell_2$ is the angle in S between ℓ_1 and ℓ_2 at A, and $\angle^\Pi F(\ell_1) \, F(A) \, F(\ell_2)$ is the angle in the plane Π between $F(\ell_1)$ and $F(\ell_2)$ at $F(A)$. Note the angle between

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[1] For the exam it is required you know the statements of all the points of Exercise 1.2, but the proofs of just points 1,2,3.

two circles in the plane is by definition the angle between the tangent lines at the intersection point. More explicitly, if two circles $C_1, C_2 \subset \Pi$ intersect at a point $P \in \Pi$, called O_1, O_2 the centers of C_1, C_2 respectively, and r_1 (resp. r_2) the unique line through P and perpendicular to PO_1 (resp. perpendicular to AO_2), define

$$\angle^{\Pi} C_1\, P\, C_2 = \angle r_1\, P\, r_2.$$

Definition 1.3 (Conformal map)**.** Given $\Omega \subset S$ or $\Omega \subset \Pi$, a map $F : \Omega \to \Pi$ is said to be *conformal*, or *angle preserving*, if it preserves angles in the sense that 6 above holds.

Note that Exercise 1.2 point 6 states that the stereographic projection is a conformal map.

Remark 1.4. We proved that every isometry is a conformal map. But, combining points 3 and 6 of Exercise 1.2 we get that the stereographic projection F is an example of a conformal map which is not an isometry.
In particular, from the point of view of the angles (i.e. the so called conformal point of view) the plane and the sphere are equivalent; instead we have seen that from the metric point of view the plane and the sphere are very different (for instance Euclid parallel postulate dramatically fails on the sphere, as every two great circles always intersect).

2 Inversion

Let $\Pi \subset \mathbb{E}^3$ be a 2-dimensional plane, let $O \in \Pi$ be a point in Π, let $S = S_R(O) \subset \mathbb{E}^3$ be a sphere in $\mathbb{E}^3$ with center O and radius R, and let $C_0 := S \cap \Pi$ be the intersection of S and Π which is a circle in Π of center O and radius R. Denote with $\mathrm{refl}_\Pi : S \to S$ be the reflection with respect to Π.

Definition 2.1 (Inversion)**.** The *inversion with respect to the circle C_0 is the map I_{C_0} :* $\Pi \setminus \{O\} \to \Pi \setminus \{O\}$ defined by $I_{C_0} := F \circ \mathrm{refl}_\Pi \circ F^{-1}$, or in other words $I_{C_0}(P) := F((\mathrm{refl}_\Pi(F^{-1}(P))))$ for every $P \in \Pi \setminus \{O\}$.

Exercise 2.2. [2]

1. *I_{C_0} maps the points outside of C_0 into the points inside of C_0 and vice-versa maps the points inside of C_0 into the points outside of C_0.*
2. *Show that $I_{C_0} : \Pi \setminus \{O\} \to \Pi \setminus \{O\}$ is a conformal map.*
   Hint: use that F is a conformal map by 6 of Exercise 1.2, refl_Π *is an isometry (so conformal), and that composition of conformal maps is a conformal map.*
3. *Show that $I_{C_0} \circ I_{C_0}$ is the identity map of $\Pi \setminus \{O\}$.*
4. *I_{C_0} maps straight lines of Π into straight lines or circles of Π; vice-versa I_{C_0} maps circles lines of Π into straight lines or circles of Π. Which circles are mapped into straight lines and which straight lines are mapped into circles?*

[2]For the exam it is required you know all the statements of Exercise 2.2, but just the proofs of points 1,2,3.